**Linear Algebra-I**
**Back-paper exam**
**Bmath First Year, AY: 24-25**

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Time: 3

Total marks: 100

Explain all the steps of your answers clearly. No marks will be awarded in absence of proper justification.

Notations: *For a matrix $A$, $\mathcal{C}(A)$ denotes the column space of $A$, $\mathcal{R}(A)$ denotes the row space of $A$, and $\mathcal{N}(A)$ denotes the null space of $A$, $r(A)$ denotes the rank of $A$.*

(1) Let $V$ and $W$ be vector spaces over a field $F$, of dimensions $n$ and $m$ respectively. Let $\mathrm{Hom}_F(V, W)$ denote the set of all linear transformations from $V$ to $W$.
(a) Show that $\mathrm{Hom}_F(V, W)$ is a vector space over $F$.
(b) Find a basis of $\mathrm{Hom}_F(V, W)$.
(c) State and prove *rank-nullity theorem* for a linear transformation $T : V \to W$. (6+6+8)

(2) Let $V$ be a vector space over a field $F$, and let $S$ be a subspace of $V$.
(a) Define complement subspace of $S$ in $V$.
(b) Let $u \in S$, and let $T$ be a complement subspace of $S$ in $V$. Define the projection of $u$ into $S$ along $T$.
(c) Show that every subspace of a vector space has a complement.
(d) Give an example of a subspace $S$ and two complements $T_1$ and $T_2$ of $S$ to show that the complements may not be unique.
(e) Consider the subspace $S = \{(x_1, x_2, x_3, x_4, x_5)^t \in \mathbb{R}^5 : \ x_1 + x_4 = 0, 2x_1 + x_3 + x_5 = 0\}$ of $\mathbb{R}^5$. Find a complement subspace $T$ of $S$ in $\mathbb{R}^5$.
(f) Let $u = (1, 1, 1, 1, 1)^t \in \mathbb{R}^5$. Find the projection of $u$ into $S$, as in (e), along the complement $T$ constructed in (e). (2+1+4+4+5+4)

(3) (a) Define column rank and row rank of a $m \times n$ matrix $A$.
(b) Prove that the column rank of $A$ is same as the row rank of $A$, (enabling us to define rank of $A$).
(c) Show that a $m \times n$ matrix $A$ has rank 1 if and only if there exist non-zero column vectors $X$ and $Y$ such that $A = XY^t$.
(d) Prove that for matrices $A$ and $B$, $r(AB) \leq \min(r(A), r(B))$. (2+6+6+6)

(4) (a) Let $A$ be a $m \times n$ matrix with rank $r \geq 1$. Define rank factorization of $A$.
(b) Show that if $(P, Q)$ is a rank-factorization of $A$, then
(i) columns of $P$ form a basis of $\mathcal{C}(A)$, and
(ii) rows of $Q$ form a basis of $\mathcal{R}(A)$.
(c) Find a rank-factorization of the matrix :

$$A = \begin{bmatrix} 1 & 3 & 1 & -1 \\ 2 & 8 & 0 & 2 \\ 1 & 2 & 2 & -3 \end{bmatrix}.$$

(2+6+6+6)

Please turn over

(5) (a) Define generalized inverse (g-inverse) of a $m \times n$ matrix A.
(b) Prove that every matrix has a g-inverse.
(c) Show that if $G$ is a g-inverse of $A$, then $AG$ is a projector into $\mathcal{C}(A)$ along $\mathcal{N}(AG)$.
(d) Find a g-inverse of

$$A = \begin{bmatrix} 1 & 3 & 4 & 1 \\ 0 & 2 & 0 & 1 \\ 1 & 0 & 0 & 2 \end{bmatrix}.$$

(2+6+8+4)

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